In designing a roller coaster there are many things to consider. How big do you want to make the first drop? How fast do you want the coaster to go? How to keep the coaster going using only gravity? What materials should it be constructed from? How much it will cost? And most importantly will your patrons have fun on the ride, and want to come back to it over and over again? Given the challenge of only designing the first rise and drop of the roller coaster (see figure 1). We are given a starting slope (line \( L_1 \)) (The ascent) that the company would like this roller coaster to build built to. They have also given us the ending slope (line \( L_2 \)) (The descent) that they want. We are to design the parabolic part in between those two points (Points P & Q) which is represented by curve (f). To be able to design this we will need to use calculus to be able to make the transitions between \( (L_1) \) to \( (f) \) at point (P), and \( (f) \) to \( (L_2) \) at point (Q). To do this we need to find the derivative of our quadratic function (curve f) which has the general form \( ax^2 + bx + c \). The reason we need to find the derivative is because the derivative represents the slope of the line tangent to the function at that specific point. In this case it will at our transition points (P & Q). The derivative is represented by lines \((L_1 \ & L_2)\). We are given the following design conditions. (See figure 1 for all descriptive references)

1. The slope of the ascent will by 0.8 Foot/Foot. Represented by line \( L_1 \)
2. The slope of the descent will be -1.6 Foot/Foot. Represented by line \( L_2 \)
3. The coaster will transition from a linear function \( (L_1) \) to a quadratic function \( (f) \), and back to a linear function \( (L_2) \)
4. The Transition to the Ascent will be at point (0,0) in a standard X/Y plane at Point (P)
5. The horizontal distance from point P to Q is 100 Feet

To find our quadratic function we start with our general form of a quadratic \( f(x) = ax^2 + bx + c \)

We need to find our derivative so we can fit the curve to our starting and ending slope. \( f'(x) = 2ax + b \)

We also need the starting linear function \( L_1(x) = 0.8x \) this gives us our starting slope.

We need to find our values for a, b, & c. Since our starting point is (0, 0) this is simplified and we set our function equal to 0 at x=0

\[
f(0) = a(0)^2 + b(0) + c = 0, \quad c = 0
\]

To find \( b \) we set the derivative equal to our starting slope at x=0. This is our starting slope we want at the transition which is 0.8ft/ft

\[
f'(0) = 2a(0) + b = 0.8 \quad b = 0.8
\]

Now that we know what \( b \) and \( c \) are we can find \( a \). We find \( a \), at where our derivative of the function transitions to our second linear function which occurs at x=100. This was the curve will be tangent to both lines \( L_1 \ & L_2 \) at points P & Q

\[
f'(100) = 2a(100) + 0.8 = -1.6 \quad a = -0.012
\]
This gives us our quadratic that is tangent to both of our given slopes at the given points.

\[ f(x) = -0.012x^2 + 0.8x \]

We want to know the height difference between the transition points so we can figure what the equation for line \( L_2 \) is. Then all we need to do is find what the function equals at \( x=0 \) & \( x=100 \). This will give us what the equation is for our second linear function, which can be found by using the point slope formula \( y - y_1 = m(x - x_1) \)

\[ f(0) = -0.012(0)^2 + 0.8(0) = 0 \quad f(100) = -0.012(100)^2 + 0.8(100) = -40 \]

\[ y - (-40) = -1.6(x - 100) \]

This gives our second linear function. \[ L_2(x) = -1.6x + 120 \]

Now that we have determined the equation for our coaster that has satisfied our given conditions, we want to graph this to make the transitions are smooth as we predicted.

We can see that our graph looks exactly as predicted. By this method we can take a starting and ending slope from any point and set our intended equation to match those slopes by taking the derivative and setting it equal the slope we want. However, we will want to refine our design since the transition from a quadratic to a linear function won’t feel smooth, even if they are tangent to our quadratic. We can fix this problem by using two cubic functions (See Figure 2) between the linear and quadratic functions. Which have the general form

\[ ax^3 + bx^2 + cx + d \]

We will change the general forms to include different variables for the new equations so that we can solve for what each variable will be by setting up a linear system, which will be shown later. First we decide that we want each equation to only be one part of the total optimized coaster design so we will give intervals (specific values of \( x \)) on which each function will play its specific part (see table below).
<table>
<thead>
<tr>
<th>Interval</th>
<th>Function</th>
<th>First Derivative</th>
<th>Second Derivative</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(-\infty, 0)$</td>
<td>$L_1(x) = 0.8x$</td>
<td>$L_1'(x) = 0.8$</td>
<td>$L_1''(x) = 0$</td>
</tr>
<tr>
<td>$(0, 10)$</td>
<td>$g(x) = kx^3 + lx^2 + mx + n$</td>
<td>$g'(x) = 3kx^2 + 2lx + m$</td>
<td>$g''(x) = 6lx + 2l$</td>
</tr>
<tr>
<td>$(10, 90)$</td>
<td>$q(x) = ax^2 + bx + c$</td>
<td>$q'(x) = 2ax + b$</td>
<td>$q''(x) = 2a$</td>
</tr>
<tr>
<td>$(90, 100)$</td>
<td>$h(x) = px^3 + qx^2 + rx + s$</td>
<td>$h'(x) = 3px^2 + 2qx + r$</td>
<td>$h''(x) = 6px + 2q$</td>
</tr>
<tr>
<td>$(100, \infty)$</td>
<td>$L_2(x) = -1.6x + 120$</td>
<td>$L_2'(x) = -1.6$</td>
<td>$L_2''(x) = 0$</td>
</tr>
</tbody>
</table>

Just as before we will need to find the derivative of the functions to make sure the slopes are equal at the transition points. We will also find the second derivative of each function as well this will allow us to make sure that the slopes all change at the same rate as each other. This will make the ride for the passengers of the roller coaster smooth and unnoticeable. We then take functions and set them equal to each other at the function value $[f(x)]$, the first derivative $[f'(x)]$, & the second derivative $[f''(X)]$ we get the following equations.

\[
L_1(0) = g(0), \quad 0 = n
\]
\[
L_1'(0) = g'(0), \quad 0.8 = m
\]
\[
L_1''(0) = g''(0), \quad 0 = 2l
\]
\[
g(10) = q(10), \quad 100k + 100l + 10m + n = 100a + 10b + c
\]
\[
g'(10) = q'(10), \quad 300k + 20l + m = 20a + b
\]
\[
g''(10) = q''(10), \quad 60k + 2l = 2a
\]
\[
q(90) = h(90), \quad 8100a + 90b + c = 729000p + 8100q + 90r + s
\]
\[
q'(90) = h'(90), \quad 180a + b = 24300p + 180q + r
\]
\[
q''(90) = h''(90), \quad 2a = 540p + 2q
\]
\[
h(100) = L_2(100), \quad 1000000p + 10000q + 100r + s = -40
\]
\[
h'(100) = L_2'(100), \quad 30000p + 200q + r = -1.6
\]
\[
h''(100) = L_2''(100), \quad 600p + 2q = 0
\]

Now if we take these equations and set them up as a system of equations (see below) we can find the values of all of our variables.

\[
\begin{array}{cccccccc}
& & & & & & & \text{n} &= 0 \\
& & & & & & & & = 0.8 \\
& & & & & & & & = 0 \\
& & & & & & & & = 0 \\
& & & & & & & & = 0 \\
-100a & -10b & -c & 1,000k & 100l & 10m & n & \text{m} = 0 \\
-20a & -b & 300k & 20l & m & \text{n} = 0 \\
-2a & 60k & 2l & \text{m} = 0 \\
8,100a & 90b & c & -729,000p & -8,100q & -90r & -s & \text{r} = 0 \\
180a & b & -24,300p & -180q & -r & \text{s} = 0 \\
2a & & -540p & -2q & \text{s} = 0 \\
1,000,000p & 10,000q & 100r & s & \text{r} = 0 \\
30,000p & 200q & r & \text{s} = 0 \\
600p & 2q & s & \text{s} = 0 \\
\end{array}
\]
Solving this system using a computer algebra system we get the following values for our variables.

<table>
<thead>
<tr>
<th>Variable</th>
<th>Exact Value</th>
<th>Decimal Approximation</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>-\frac{1}{75}</td>
<td>-0.0133</td>
</tr>
<tr>
<td>b</td>
<td>\frac{14}{15}</td>
<td>0.9333</td>
</tr>
<tr>
<td>c</td>
<td>-\frac{4}{9}</td>
<td>-0.4444</td>
</tr>
<tr>
<td>k</td>
<td>-\frac{1}{2250}</td>
<td>-0.0004</td>
</tr>
<tr>
<td>l</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>m</td>
<td>\frac{4}{5}</td>
<td>0.8</td>
</tr>
<tr>
<td>n</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>p</td>
<td>\frac{1}{2250}</td>
<td>0.0004</td>
</tr>
<tr>
<td>q</td>
<td>-\frac{2}{15}</td>
<td>0.1333</td>
</tr>
<tr>
<td>r</td>
<td>\frac{176}{15}</td>
<td>11.7333</td>
</tr>
<tr>
<td>s</td>
<td>-\frac{2920}{9}</td>
<td>324.4444</td>
</tr>
</tbody>
</table>

Plugging the values for the variables back into the general forms from the previous table gives us the optimized design equations for our roller coaster, over each equation’s specified interval.

1. \( L_1(x) = 0.8x \), \((-\infty, 0)\)

2. \( g(x) = -\frac{1}{2250} x^3 + \frac{4}{5} x \), \((0,10)\)

3. \( q(x) = -\frac{1}{75} x^2 + \frac{14}{15} x - \frac{4}{9} \), \((10,90)\)

4. \( h(x) = \frac{1}{2250} x^3 - \frac{2}{15} x^2 + \frac{176}{15} x - \frac{2920}{9} \), \((90,100)\)

5. \( L_2(x) = -1.6x + 120 \), \((100,\infty)\)

Now we can verify graphically that our optimized design is smoother and will overall end up being a much better ride for the passengers.
So now we can see that mathematics especially calculus is very powerful in the things that make our life better and easier. By just taking derivatives we can find the slope of any equation along any point and combine functions between multiple points to create smooth transitions between them. This is especially useful when making roller coasters. The research involved in making this project I learned how powerful calculus really is. So many things in this world can be explained by using the basic principles of calculus, in this case differential calculus. I hope to convey to you why this is important. The higher level mathematics and sciences open up a world of creativity. Things that you may have always wanted to do but don’t necessarily know the way to do it can be explained by higher math and can give you the skills to open up your imagination and create the things of your dreams. When Isaac Newton came up with calculus, and then physics I’m sure he never fathomed that we could be where we are today.

For me personally I understand to some extend how that with mathematics so many of the world’s problems can be solved by applying its principles. From minimizing costs, and maximizing profits in the business world. To designing rockets and sending men, and women into space. The power to create solutions to the world’s biggest problems like a clean renewable source of energy can all be solved with math. I know that as I have started learning the higher math and sciences my imagination has been expanded many times, and as a consequence my dreams have only become larger. I hope that from reading this you will be inspired to tackle the higher math and sciences. Even though they are very challenging courses from my experience the work put into these classes have returned far more positive benefits than I originally imagined.